A Multi-Factor Bottom-Up Model for Pricing Credit Derivatives

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ABSTRACT

In this paper we continue the study of the stress event model, a simple and intuitive dynamic model for credit risky portfolios, proposed by Duffie and Singleton (1999). The model is a bottom-up version of the multi-factor portfolio credit model proposed by Longstaff and Rajan (2008). By a novel identification of independence conditions, we are able to decompose the loss distribution of a credit risky portfolio into a series expansion which not only provides a clear picture of the characteristics of the loss distribution but also suggests a fast and accurate approximation for it. Our approach has three important features: (i) it is able to match standard CDS index tranche prices and the underlying CDS spreads, (ii) its computational speed is very fast, comparable to that of the Gaussian copula model, (iii) the computational cost for additional factors is mild, allowing for more flexibility for calibrations and opening the possibility of studying multi-factor default dependence of a portfolio via a bottom-up approach. We demonstrate the tractability and efficiency of our approach by calibrating a three-sector model to investment grade CDS index tranches.

keywords: credit derivatives, CDO, bottom-up approach, multi-name, intensity-based, risk and portfolio.

1. Introduction

The bottom-up stress event model, proposed by Duffie and Singleton (1999), is a simple and intuitive model for portfolio credit risk. The model is seldom applied in practice since it is generally believed that the default times, as well as the loss distribution, of a portfolio under this modeling framework can only be generated by computationally expensive Monte Carlo simulation. In this paper an alternative approach is taken, avoiding Monte Carlo simulations, making the model tractable and leading to an efficient calibration to market data. The idea of the stress event model is easy to understand. Besides idiosyncratic default, each firm may default if there is a joint credit event (Duffie and Singleton 1999) or alternatively referred to as a stress event (Schönbucher 2003). This allows default correlation through both changes in stress event intensities as well as the occurrences of stress events. The formal definition of the default time of a firm is given in Section 3. In Section 4, we develop a new approach to compute the loss distribution of a portfolio for the stress event model. We first identify independence conditions under which defaults of firms are independent. The loss distribution can then be decomposed into a series expansion for which each term admits a closed form expression. It turns out that only the first few terms of the series are needed to accurately approximate the loss distribution since stress events are infrequent. This leads to a very efficient method to compute the loss distribution of a portfolio.

The analysis by Longstaff and Rajan (2008) suggest that risk-neutral default dependence in a typical portfolio of corporate credits may involve multiple factors. The stress event model, like other bottom-up models, faces significant computational challenges when the number of non-idiosyncratic factors is more than one. This curse of dimensionality comes from the rapid increase of the number of conditional loss distributions needed to compute the loss distribution. For example, if the number of conditional loss distributions needed to compute in a one-factor model is 100, it is expected that the number of conditional loss distributions needed in a L-factor model would be 100^L. This is not the case for our new approach due to the novel identification of independence conditions which result in important simplifications to the corresponding series expansion of a loss distribution for the stress event model. It turns out that the number of conditional loss distributions needed in our approach only increases mildly with the number of non-idiosyncratic factors. Hence, the increase in computational time due to additional non-idiosyncratic factors in the stress event model is much smaller than that in other bottom-up models. This extra flexibility for adding additional non-idiosyncratic factors in the stress event model leads to a better fit to market data.

We demonstrate the tractability and efficiency of our approach by two calibration examples in Section 5. In the first example, the model is calibrated to the first five tranches of the 5-year CDX.NA.IG series 13 and the 125 underlying CDS spreads simultaneously. CDS spreads are matched exactly and model implied tranche prices are within the bid-ask spreads. In the second calibration example, we regard the stress event model as a top-down model and calibrate it to the term structure of the Markit iTraxx Europe series 7 on four different days simultaneously. When calibrating the 26 parameters of the model to the 60 data points, the root-mean-square relative error of the model implied tranche prices is 4.08%.

2. Related Literature

There are two approaches in multi-name credit risk modeling. In the bottom-up approach, individual losses of names are modeled and then aggregated over the portfolio. This approach is pursued by Duffie and Singleton (1999), Duffie and Gârleanu (2001), Mortensen (2006), Joshi and Stacey (2006), Papageorgiou and Sircar (2007), Peng and Kou (2008), Eckner (2009) and others. On the other hand, the top-down approach, which models the dynamics of a portfolio loss distribution directly, is also an active research area. Top-down models are investigated by Errais et al. (2006), Brigo et al. (2007), Cont and Minca (2008), Longstaff and Rajan (2008), Arnsdorf and Halperin (2008), Bayraktar and Yang (2009), Giesecke et al. (2010) and others.

3. Model Formulation

For notational consistency, we reserve the subscript index i for specifying a firm and the superscript l for indexing a sector in the rest of this paper. In a portfolio which consists of credit risky securities issued by N firms, the default time of firm i under the stress event model framework is defined as follows:

$$\tau_i = \inf\left\{ s \geq 0 : \bar{N}_i(s) + \sum_{l=1}^{L} \sum_{j=1}^{\infty} \mathbf{1}_{\{s > t_j^l\}} X_{i,j}^l > 0 \right\}, \tag{1}$$

for $i = 1, .., N$, where

- t_j^l is the j-th jump time of a Poisson process $N^l(s)$ associated with sector l,
- all $\bar{N}_i$ and N^l are independent Poisson processes with intensities $\bar{\lambda}_i(s)$ and $\lambda^l(s)$ respectively,
- $\mathbf{1}_{\{s > t_j^l\}}$ is an indicator function that equals one if $s > t_j^l$ and zero otherwise,
- $X_{i,j}^l$ are Bernoulli random variables indicating if a stress event at time t_j^l has killed the i-th firm or not, independent of the Poisson processes,
- L is the number of non-idiosyncratic factors affecting a portfolio; we will interchangeably use the terms "non-idiosyncratic factor" and "sector" since firms affected by a common non-idiosyncratic factor can be considered belonging to a common sector.

$\bar{N}_i$ is an idiosyncratic Poisson process associated with firm i which is driven by firm-specific factors. Once there is a jump in $\bar{N}_i$, firm i defaults immediately. In addition, if N^l has a jump at t^l_j, firm i may default with a probability $\mathrm{P}(X^l_{i,j} = 1) = p^l_i$. We say that firm i's default can be caused by the l-th sector if $p^l_i > 0$. It is worth noting that only the first jump in $\bar{N}_i$ is relevant for the default triggering of the i-th firm and later jumps are irrelevant, whereas each jump in N^l could be a default triggering event.

The Poisson processes $\bar{N}_i$ and N^l considered in this paper are doubly stochastic processes, i.e. the intensities $\bar{\lambda}_i$ and λ^l may also be stochastic. In the general exposition of the model, it is not necessary to specify the processes followed by the intensities. In Section 5, where the model is calibrated to data, the intensities will be taken to be constant in one case and follow an affine-jump diffusion process in another.

4. Loss Distribution

The loss of a portfolio is a dynamic process which evolves stochastically over time. A common approach for calculating the loss distribution of a credit risky portfolio in bottom-up models is by computing the loss distribution under conditional independence. The unconditional default distribution is then the weighted sum of the conditional ones, i.e.

$$\mathrm{P}(D(t) = n) = \int_\Omega \mathrm{P}(D(t) = n|\omega)\mathrm{P}(d\omega), \quad n = 1, ..., N, \tag{2}$$

where $D(t)$ is the number of defaults by time t and ω is a condition under which defaults of firms are independent. We assume that the recovery rate of each security is a constant R

and a uniform notional amount δ for all firms in the portfolio, thus the loss of a portfolio is

$$L_t = \sum_{i=1}^{N} \delta_i (1 - R_i) \mathbf{1}_{\{\tau_i \le t\}} = \delta(1 - R) \sum_{i=1}^{N} \mathbf{1}_{\{\tau_i \le t\}} = \delta(1 - R) D(t). \tag{3}$$

Therefore, modeling the loss distribution is equivalent to modeling the default distribution. The first challenge of evaluating Eq.(2) is to find a computationally efficient scheme to calculate the conditional loss distribution $\mathrm{P}(D(t) = n|\omega)$. To this end, we adopt the recursive algorithm suggested by Andersen et al. (2003). In fact, the recursive algorithm can also compute the loss distribution of a portfolio with different recovery rate and notional for each name. The computational cost for each conditional loss distribution is relatively expensive for a large portfolio, thus the number of conditional loss distributions needed to compute the unconditional loss distribution for each time t significantly affects the time of the overall calculation. It turns out that only a moderate number of conditional loss distributions are need to accurately approximate the full loss distribution in our approach. The second challenge lies in the evaluation of $\mathrm{P}(d\omega)$. This is in fact a threefold challenge. One needs to identify conditions under which defaults are independent, choose a partition for the probability space Ω that is as small as possible, and evaluate the probabilities of these independence conditions. We present a novel identification of independence conditions which arises naturally from the formulation of the stress event model. We also introduce a systematic way of choosing a countable partition of Ω which automatically arranges the sizes of $\mathrm{P}(d\omega)$ in descending order. In addition, we provide explicit formulas for the probabilities of independence conditions for a wide class of stochastic intensities.

a. Independence Conditions

For intensity-based models, like the correlated intensity model by Duffie and Gârleanu (2001), a realization of the non-idiosyncratic part of firms' default intensities is usually employed as an independence condition for defaults. We believe that this framework can provide a similar set of independence conditions for the stress event model. However, this approach may not be very efficient in the present situation and will not be pursued here. Instead, we follow a different approach to identifying independence conditions for the stress event model, which will prove to lead to a more efficient analysis and calculation. These independence conditions arise naturally from the definition of the individual firm's default times in the stress event model as they are related to the occurrences of the non-idiosyncratic events in the model. Consider a scenario characterized by non-idiosyncratic events

$$\omega^u = \omega(u(L, \vec{m}_L, t)) = \left\{\omega : t_j^l(\omega) = u_j^l \in (0, t],\ j = 1, ..., m_l,\ l = 1, ..., L\right\}, \tag{4}$$

where

$$u(L, \vec{m}_L, t) = \begin{pmatrix} u_1^1 & u_2^1 & \dots & \dots & u_{m_1}^1 & \\ u_1^2 & u_2^2 & \dots & u_{m_2}^2 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_1^L & u_2^L & \dots & \dots & \dots & u_{m_L}^L \end{pmatrix} \quad \text{and} \quad \vec{m}_L = (m_1, m_2, ..., m_L). \tag{5}$$

$u(L, \vec{m}_L, t)$ is an array of L rows and each row has m_l entries which specifies the jump times of N^l up to time t. This is the scenario that there are m_l stress events occurring at $u_1^l, u_2^l, ..., u_{m_l}^l$ in increasing order all before time t in the l-th sector for $l = 1, ..., L$. For a given ω^u, Eq.(1) becomes

$$\tau_i(\omega^u) = \inf_{s \geq 0} \left\{ \bar{N}_i(s) + \sum_{l=1}^{L} \left(\sum_{j=1}^{m_l} \mathbf{1}_{\{s > u_j^l\}} X_{i,j}^l + \sum_{j=m_{l+1}}^{\infty} \mathbf{1}_{\{s > t_j^l\}} X_{i,j}^l \right) > 0 \right\}. \qquad (6)$$

Eq.(6) is just a splitting of the terms in the defining Eq.(1), under the condition ω^u, into the terms before t where the occurrence times of stress events are known and after t where they are random. Define

$$\tilde{\tau}_i(\omega^u) = \inf_{s \geq 0} \left\{ \bar{N}_i(s) + \sum_{l=1}^{L} \sum_{j=1}^{m_l} \mathbf{1}_{\{s > u_j^l\}} X_{i,j}^l > 0 \right\}, \qquad (7)$$

which is almost identical to Eq.(6) except that the last sum inside the brackets is deleted. Note that if $\tau_i(\omega^u) \leq t$, then

$$\tau_i(\omega^u) = \tilde{\tau}_i(\omega^u), \qquad (8)$$

since a default must be triggered by a jump of $\bar{N}_i$ or N^l before t and is irrelevant to anything that happens after t. The default indicators under ω^u are

$$\mathbf{1}_{\{\tau_i(\omega^u) \leq t\}} = \mathbf{1}_{\{\tilde{\tau}_i(\omega^u) \leq t\}}, \qquad (9)$$

for $i = 1, ..., N$. The key observation leading to an independence condition is that $\mathbf{1}_{\{\tilde{\tau}_i(\omega^u) \leq t\}}$ are independent since all $\tilde{\tau}_i(\omega^u)$ are defined by independent Poisson processes and Bernoulli random variables as indicated by Eq.(7). Consequently, we can apply the recursive algorithm of Andersen et al. (2003) to compute the conditional loss distribution of a portfolio.

b. Conditional Individual Survival Probability and Conditional Loss Distribution

In order to build a conditional loss distribution, we have to compute the individual survival probability for each firm under an independence condition. The conditional survival probability of firm i for a given ω^u as specified by Eq.(4) is

$$\mathrm{P}\left(\tau_i > t|\omega^u\right) = \mathrm{P}\left(\bar{\tau}_i > t|\omega^u\right) \prod_{l=1}^{L} \mathrm{P}\left(\tau_i^{l,X} > t|\omega^u\right), \tag{10}$$

where $\bar{\tau}_i$ is the first jump time of the idiosyncratic Poisson process $\bar{N}_i$ and $\tau_i^{l,X}$ is a jump time t_j^l of N^l such that it is the first time the Bernoulli random variable $X_{i,j}^l = 1$ among $j = 1, 2, ...$ The seeming notational inconsistency between the i and j in the random stopping times $\tau_i^{l,X}$ and t_j^l arises from the effort to make precise that among the stress events in the l-th sector, $\tau_i^{l,X}$ is the first time that affects the i-th firm through the random variable $X_{i,j}^l$. Since the idiosyncratic default intensity $\bar{\lambda}_i$ does not depend on the occurrences of the stress events in the sectors,

$$\mathrm{P}\left(\bar{\tau}_i > t|\omega^u\right) = \mathrm{P}\left(\bar{\tau}_i > t\right) \tag{11}$$

$$= \mathrm{E}\left[e^{-\int_0^t \bar{\lambda}_i(s)ds}\middle|\bar{\lambda}_i(0)\right]. \tag{12}$$

On the other hand,

$$\mathrm{P}\left(\tau_i^{l,X} > t|\omega^u\right) = (1 - p_i^l)^{m_l} \tag{13}$$

is the conditional survival probability that firm i is not killed by the m_l stress events in the l-th sector before t. As a result,

$$\mathrm{P}\left(\tau_i > t|\omega^u\right) = \mathrm{E}\left[e^{-\int_0^t \bar{\lambda}_i(s)ds}\middle|\bar{\lambda}_i(0)\right] \prod_{l=1}^{L}(1 - p_i^l)^{m_l}. \tag{14}$$

It is important to note that this conditional survival probability as well as the corresponding conditional loss distribution depend only on idiosyncratic intensities and, most crucially, the number of stress events in each sector by time t, but NOT the occurrence times of the stress events. Consequently,

$$\mathrm{P}\left(\tau_i > t|\omega^u\right) = \mathrm{P}\left(\tau_i > t|\vec{m}_L\right), \tag{15}$$

and the conditional loss distribution becomes

$$\mathrm{P}(D(t) = n|\omega^u) = \mathrm{P}(D(t) = n|\vec{m}_L), \tag{16}$$

which can be computed by using the conditional survival probabilities given by Eq.(14). Eq.(16) makes a crucial point that although there are uncountable independence conditions ω^u, the number of conditional loss distributions is countable since the number of possible scenarios of stress events, specified by $\vec{m}_L = (m_1, m_2, ..., m_L)$, is countable.

c. Unconditional Loss Distribution for Deterministic Intensities

For a deterministic intensity λ^l, the probability of m_l stress events occurring by time t in the l-th sector is

$$\mathrm{P}(m_l \text{ stress events occur until time } t) = e^{-\Lambda^l(t)} \frac{(\Lambda^l(t))^{m_l}}{m_l!}, \tag{17}$$

where

$$\Lambda^l(t) = \int_0^t \lambda^l(s) ds \tag{18}$$

is the cumulative intensity. Furthermore, the probability that the $m_l > 0$ stress events occur at u_1^l,...,$u_{m_l-1}^l$ and $u_{m_l}^l$ is

$$\mathrm{P}(m_l \text{ stress events at } u_1, u_2, ..., u_{m_l}) = e^{-\Lambda^l(t)} \prod_{j_{m_l}=1}^{m_l} \lambda^l(u_{j_{m_l}}^l) du_{j_{m_l}}^l. \tag{19}$$

It can be shown that the full loss distribution is

$$\mathrm{P}(D(t) = n) = \sum_{k=0}^{\infty} \phi_k(t; n), \tag{20}$$

where

$$\phi_k(t;n) = \sum_{\sum m_l = k} \left(\prod_{l=1}^{L} \frac{1}{m_l!} \int_{(0,t]^{m_1 \times \cdots \times m_L}} \mathrm{P}(D(t) = n | \omega^u) \mathrm{P}(d\omega^u) \right) \tag{21}$$

is defined as the k-th component of the loss distribution. It is worth noting that the domain of integration in Eq.(21) should be a multi-dimensional simplex since $u_1^l < u_2^l < \cdots < u_{m_l}^l$ for each l. By symmetry, we extend the domain of integration to a multi-dimensional box $(0,t]^{m_l}$ and divide the integral by $m_l!$ for each l, which yields the current form of Eq.(21). We can simplify Eq.(21) as follows:

$$\begin{aligned}
\phi_k(t;n) &= \sum_{\sum m_l = k} \left(\int_{(0,t]^{m_1 \times \cdots \times m_L}} \mathrm{P}(D(t) = n | \omega^u) \prod_{l=1}^{L} \frac{e^{-\Lambda^l(t)}}{m_l!} \prod_{j_{m_l}=1}^{m_l} \lambda^l(u^l_{j_{m_l}}) du^l_{j_{m_l}} \right) && (22) \\
&= \sum_{\sum m_l = k} \left(\mathrm{P}(D(t) = n | \vec{m}_L) \int_{(0,t]^{m_1 \times \cdots \times m_L}} \prod_{l=1}^{L} \frac{e^{-\Lambda^l(t)}}{m_l!} \prod_{j_{m_l}=1}^{m_l} \lambda^l(u^l_{j_{m_l}}) du^l_{j_{m_l}} \right) && (23) \\
&= \sum_{\sum m_l = k} \left(\mathrm{P}(D(t) = n | \vec{m}_L) \prod_{l=1}^{L} \frac{e^{-\Lambda^l(t)}}{m_l!} \left(\Lambda^l(t) \right)^{m_l} \right). && (24)
\end{aligned}$$

In the above derivation, we utilize the property of a conditional loss distribution that it does not depend on the occurrence times of the stress events but depends only on the number of stress events in each sector (see Eq.(16)). We call $\phi_k(t;n)$ the k-th order term of the unconditional loss distribution. It is important to notice that $\phi_k(t;n)$, as a countable sum in Eq.(24), is a significant simplification of its original form in Eq.(22) which is a sum of multi-dimensional integrals. This simplification is the crux leading to an efficient algorithm for the unconditional loss distribution.

d. Unconditional Loss Distribution for Stochastic Intensities

For stochastic sector intensities λ^l, the loss distribution contributed by exactly k stress events altogether by t can be computed by taking the expectation of Eq.(24) over all possible intensity paths of λ^l. Specifically,

$$\phi_k(t;n) = \sum_{\sum m_l = k} \left(\mathrm{E}\left[\mathrm{P}(D(t)=n|\vec{m}_L) \prod_{l=1}^{L} \frac{e^{-\Lambda^l(t)}}{m_l!} \left(\Lambda^l(t)\right)^{m_l} \middle| \lambda^1(0), ..., \lambda^L(0) \right] \right) \tag{25}$$

$$= \sum_{\sum m_l = k} \left(\mathrm{P}(D(t)=n|\vec{m}_L) \prod_{l=1}^{L} \frac{1}{m!} \mathrm{E}\left[e^{-\Lambda^l(t)} \left(\Lambda^l(t)\right)^{m_l} \middle| \lambda^l(0) \right] \right). \tag{26}$$

The conditional loss distribution $\mathrm{P}(D(t) = n|\vec{m}_L)$ is independent of the intensities λ^l and can be constructed by the conditional survival probabilities Eq.(14). The expectation

$$\mathrm{E}\left[e^{-\Lambda^l(t)} \left(\Lambda^l(t)\right)^{m_l} \middle| \lambda^l(0) \right] \tag{27}$$

admits a closed form expression for a wide class of stochastic processes. We provide an explicitly expression of Eq.(27) when λ^l is an affine-jump diffusion process in Appendix A. Consequently, introducing stochastic sector intensities λ^l in the stress event model does not undermine the tractability of the model, and Eq.(26) can be computed as easily as Eq.(24). Define

$$|\phi_k(t)| = \mathrm{P}(\text{exactly } k \text{ stress events occur until time } t) \tag{28}$$

$$= \sum_{n=0}^{N} \phi_k(t;n) \tag{29}$$

$$= \sum_{\sum m_l = k} \left(\prod_{l=1}^{L} \frac{1}{m_l!} \mathrm{E}\left[e^{-\Lambda^l(t)} \left(\Lambda^l(t)\right)^{m_l} \middle| \lambda^l(0) \right] \right), \tag{30}$$

which measures the contribution of the k-th order term to the loss distribution. Since the intensity λ^l of each sector is generally quite small, $|\phi_k(t)|$ is negligible for large k. Consequently, only the first few loss distribution components $\phi_k(t;n)$ are necessary to construct the full loss distribution and this leads to an efficient approximation for it. Furthermore, define

$$\epsilon_K(t) = \mathrm{P}(\text{total number of stress events by } t > K) \tag{31}$$

$$= 1 - \sum_{k=0}^{K} |\phi_k(t)|, \tag{32}$$

which is a measure of the error of the K-th order approximation for a loss distribution. $\epsilon_K(t)$ is the probability of scenarios that are not considered in the K-th order approximation. The closer the value ϵ_K is to zero, the more accurate is the approximation.

e. Approximation of the Loss Distribution

In order to prevent the leak of probability $\epsilon_K(t)$ growing over time due to the finite order approximation to a loss distribution, we add the unaccounted probability to the highest order term in the calculation such that the updated unconditional loss distribution of the K-th order term is

$$\tilde{\phi}_K(t;n) = \left(\frac{1 - \sum_{k=0}^{K-1} |\phi_k(t)|}{|\phi_K(t)|} \right) \phi_K(t;n), \tag{33}$$

and approximate the full loss distribution as

$$\mathrm{P}(D(t) = n) \approx \sum_{k=0}^{K-1} \phi_k(t;n) + \tilde{\phi}_K(t;n). \tag{34}$$

Hence, the total probability of the loss distribution approximation is one for all t.

On the other hand, the conditional loss distribution under each scenario $\vec{m}_L$ is a multinomial distribution with individual default probability $1 - \mathrm{P}(\tau_i > t|\vec{m}_L)$, where the conditional survival probability is given by Eq.(14). The modified Andersen Sidenius Basu (mASB) algorithm proposed by Eckner (2009) can speed up the the computation of conditional loss distributions. Besides, the Gaussian approximation by Shelton (2004) and the Adjusted Binomial Approximation by O'Kane (2008) also provide efficient means to approximate conditional loss distributions. These approximations can usually help to compute good estimates of the model parameters with small relative errors. In calibrations to market data, a good estimate of the parameter set can be obtained by using approximation schemes like Gaussian or Adjusted Binomial approximation in the optimization. One then uses this estimate as an initial guess for the optimization which adopts the more accurate algorithm.

f. Efficiency Analysis

The bottleneck of the computation of the loss distribution for bottom-up models is usually the calculation of conditional loss distributions. In the recursive algorithm proposed by Andersen et al. (2003) or the modified one by Eckner (2009), the number of calculations needed to compute conditional loss distributions, $\mathrm{P}(D(t) = n|\omega)$ where $n = 0, 1, ..., N$, for a portfolio with N names is proportional to N^2 . The typical number of firms N is quite large (usually over 100), which makes the computation of conditional loss distributions relatively expensive.

In the stress event model, the number of conditional loss distributions, $\mathrm{P}(D(t) = n|\vec{m}_L)$, needed to compute the loss distribution in the K-th order approximation with L sectors is equivalent to the number of solutions to the following Diophantine inequality

$$m_1 + m_2 + \cdots + m_L \leq K, \tag{35}$$

or equivalently the total number of solutions to the following $K+1$ Diophantine equalities

$$m_1 + m_2 + \cdots + m_L = k, \quad k = 0, 1, ..., K, \tag{36}$$

which is

$${}_{L+K}C_K = \frac{(L+K)!}{L!K!}. \tag{37}$$

Table 1 shows the number of conditional loss distributions needed for different values of L and K. The order K of the approximation in Eq.(34) should be dynamically determined via Eq.(32) depending on the required accuracy. In the two calibrations that we will discuss later, we find that using $K = 8$ and $K = 5$ can approximate the loss distributions accurately in the two-sector and three-sector models respectively which has 45 and 56 conditional loss distributions respectively. For the one-factor Gaussian copula, the typical number of conditional loss distributions is 50, so our approach has a computational speed similar to that of the Gaussian copula.

On the other hand, Table 1 shows that the number of conditional loss distributions needed increases mildly with L, so the additional computational cost for more non-idiosyncratic factors is moderate. This is not the case for most bottom-up models where the number of conditional loss distributions needed to compute the loss distribution increases rapidly with the number of non-idiosyncratic factors. The moderate cost for additional non-idiosyncratic factors gives our model more flexibility to match market data. In addition, it opens up the possibility to study multi-factor default dependence in a portfolio via a bottom-up approach.

5. Calibration

a. Calibration to CDX.NA.IG Series 13 Tranches and Underlying CDS Spreads

We calibrate the three-sector ($L = 3$) stress event model to market data. The data set contains the first five index tranche prices of CDX.NA.IG series 13 and the 125 underlying CDS spreads on April 15 2010 which were obtained from a Bloomberg terminal. All tranches and CDSs are 5-year contracts. The quotes of the index tranches and the statistics of the CDS spreads are shown in Table 2 and Table 3 respectively. We assume a constant recovery rate $R = 0.35$ which is consistent with empirical evidence for senior unsecured bonds reported by Hamilton et al. (2004). Furthermore, risk-free interest rates are taken from swap curves. We assume time-independent intensities for all the Poisson processes. Consequently, the default intensity for each firm i can be computed by the so-called credit triangle (O'Kane 2008), i.e.

$$\lambda_i = \frac{S_i}{(1-R)}, \quad i = 1, ..., N, \tag{38}$$

where S_i is the 5-year CDS spread of firm i. Hence, $S_i/(1-R)$ imposes a constraint for other parameters in the default intensity of firm i as follows:

$$\frac{S_i}{(1-R)} = \lambda_i = \bar{\lambda}_i + p_i^1\lambda^1 + p_i^2\lambda^2 + p_i^3\lambda^3. \tag{39}$$

This model specification has $4N+3$ parameters and N constraints. We favor a parsimonious model which is flexible to match tranche spreads. Specifically, we choose a parameter set of six members

$$\Theta = \{\lambda^1, p^1, \lambda^2, p^2, \lambda^3, p^3\}, \tag{40}$$

for the calibration, where λ^l are the stress event intensities and p^l are representative impact factors. We define the relative credit quality c_i of a firm as its CDS spread divided by the average CDS spread of the portfolio. We then set $p_i^l = \min\{c_i p_i^l, 1\}$ and compute $\bar{\lambda}_i$ by Eq.(39). However, $\bar{\lambda}_i$ could be negative for some cases and we have to rescale p_i^l to make it non-negative. The detailed specifications of p_i^l and $\bar{\lambda}_i$ in terms of the parameters in Θ are provided in Appendix B. We use the root-mean-square error

$$\text{RMSE} = \sqrt{\frac{1}{5}\sum_{j=1}^{5}\left(\frac{\tilde{S}_{tr,j} - S_{tr,j}}{S_{tr,j}^{Bid} - S_{tr,j}^{Ask}}\right)^2} \tag{41}$$

as the objective function in this calibration, where $S_{tr,j}$, $S_{tr,j}^{Bid}$ and $S_{tr,j}^{Ask}$ are the market mid, bid and ask of the j-th tranche respectively, and $\tilde{S}_{tr,j}$ are the model implied tranche prices. The parameter set Θ which minimizes Eq.(41) is presented in Table 4. Table 5 shows the model implied tranche prices. As can be seen, all of them are within bid-ask spreads. The MATLAB implementation takes about 0.2 seconds for pricing all tranches using a fixed set of parameters[1]. Since the model has six non-idiosyncratic parameters to match five tranche prices, the good model fit is not surprising at first sight. However, since model implied tranche prices depend in a nonlinear fashion on the model parameters, and intensities λ^l have to be non-negative and conditional default probabilities p^l have to be between 0 and 1, it is not clear a prior which tranche price combinations the model can match.

In the empirical study of Longstaff and Rajan (2008), CDX tranches are priced as if losses of 35%, 6% and 0.4% of the portfolio occur with expected frequencies of 763, 41.5 and 1.2 years, respectively. One can consider the 35% loss with frequency 763 years as a

[1]Using a personal laptop computer with Intel(R) CPU T2050 1.60 GHz, 1.49 GB RAM.

result from global crises, the 6% loss with frequency of 41.5 years as a result from sectoral credit events, and the 0.4% loss with frequency 1.2 years as a result due to idiosyncratic defaults. According to the calibrated parameters of our model in Table 4, we find that $\lambda^3 = 0.0038$ with $p^3 = 1$ which corresponds to global crises that cause an expected loss of 65% of the portfolio with expected frequency of 265 years. The two Poisson processes with intensities $\lambda^1 = 0.043$ and $\lambda^2 = 0.013$ both correspond to the sectoral credit events of different magnitudes that cause an expected loss of 6.22% with expected frequency 17.8 years. We see that both frequencies of the global and the sectoral credit event implied by our parameters are higher than those found by Longstaff and Rajan (2008). The impacts of the credit events, i.e. the expected losses, are also higher in our case. The higher frequencies and impacts in our case are not surprising, as our parameters are calibrated to post-crisis market data whereas the parameters computed by Longstaff and Rajan (2008) are calibrated to pre-crisis market data. Although the expected losses and frequencies are quite different, they agree qualitatively. In both cases, the market expects a global crisis with frequency in the range of centuries which causes a loss over 30% of the portfolio, and a sectoral credit event with frequency in the range of decades which causes a loss about 6%. The average idiosyncratic intensity implied by the calibrated parameters is 0.0044 which corresponds to a loss of 0.52% with expected frequency 1.83 years. These values are similar to those found by Longstaff and Rajan (2008).

b. Calibration to the Term Structure of iTraxx Europe Tranches on Multiple days

One of the main merits of our approach is that introducing stochastic intensities to the model does not undermine the tractability and efficiency. We will apply the stress event model as a top-down model in this subsection, i.e. the model is calibrated to index tranches only. The data that we are using for the calibration are obtained from the Monthly Markit iTraxx Tranche Fixings (see www.creditfixings.com). They consists of four days of market data of the Markit iTraxx Europe series 7 observed on March 30, April 30, May 31 and June 29 in 2007. On each day, there are five standard tranches with maturities 5, 7 and 10 years. There are altogether 60 data points and they are shown in Table 6. We employ the two-sector stress event model ($L = 2$) in the current calibration. Since a parsimonious parameter set is favored, we assume that the idiosyncratic default intensity of each firm follows the same dynamics of a stochastic process and that the probabilities of default given a stress event are the same for all firms, i.e. $p_i^1 = p^1$ and $p_i^2 = p^2$ for all i. There are altogether three different types of intensity processes in this specification, one for the idiosyncratic factor and two for the non-idiosyncratic factors. We further assume that each type follows an affine jump-diffusion process (Duffie et al. 2000)

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dB_t + dJ_t, \quad \lambda_t = \lambda_0, \tag{42}$$

with the mean reverting level $\theta = 0$. A brief discussion of this affine-jump diffusion process is presented in Appendix A. Recall that $\Lambda(t) = \int_0^t \lambda_s ds$ and

$$\mathrm{E}\left[e^{-\Lambda(t)}\frac{(\Lambda^k(t))^k}{k!}\Big|\lambda_0\right], \tag{43}$$

which is the probability that there are k stress events in a sector, admits a closed form expression for an affine jump-diffusion process of type (42). We report the closed form expression of Eq.(43) for $k = 0$ and derive expressions for positive integers k in Appendix A.

There are four parameters in each intensity process and two constant impact factors p^1 and p^2, thus the current model specification has 14 parameters that are not time-varying:

$$\Theta_{\text{fix}} = \{\bar{\kappa}, \bar{\sigma}, \bar{l}, \bar{\mu}, \kappa^1, \sigma^1, l^1, \mu^1, p^1, \kappa^2, \sigma^2, l^2, \mu^2, p^2\}. \tag{44}$$

Besides, there are three initial intensities for the affine-jump diffusion processes on each of the four days. Consequently, we are calibrating 26 model parameters to 60 data points. Similar to the previous calibration example, a constant recovery rate $R = 0.35$ is used in the calibration and risk-free interest rates are taken from swap curves.

The model parameters are calibrated by minimizing the root-mean-square of the relative error

$$\text{RMSE} = \sqrt{\frac{1}{60}\sum_{l=1}^{4}\sum_{k=1}^{3}\sum_{j=1}^{5}\left(\frac{\tilde{S}_{tr,j}^{T_k,t_l} - S_{tr,j}^{T_k,t_l}}{S_{tr,j}^{T_k,t_l}}\right)^2}, \tag{45}$$

where $T_1 = 5$, $T_2 = 7$ and $T_3 = 10$ are the maturities of the tranches, t_l is the index for the observing date and j is the index for the tranche. Thus, $S_{tr,j}^{T_k,t_l}$ is the price of the j-th tranche with maturity T_k observed on t_l and $\tilde{S}_{tr,j}^{T_k,t_l}$ is the corresponding model implied tranche price.

We employ an eighth order ($K = 8$) approximation, and the calibrated parameters and the model implied tranche prices are presented in Table 7 and Table 8 respectively. The model implied tranche prices match quite well with the market mid prices in general with the root-mean-square relative error RMSE $= 4.08\%$ and the maximum relative error 10.4%. It is worth noting that we use the same parameter set Θ_{fix} for all days while changing three initial intensities $\bar{\lambda}_0$, λ_0^1 and λ_0^2 to obtain a reasonably good fit to 15 data points on each day. We implemented the calibration using MATLAB and the same computer as in the first calibration example. It takes about 0.8 second for each pricing (compute 60 tranche prices for each set of parameters). We see that all the default intensities are explosive, i.e. the risk-neutral mean reverting rates $\bar{\kappa}$, κ^1 and κ^2 are negative. The negative mean reverting rates are necessary for matching the steep term structure of tranche spreads. In the calibrations of the correlated intensity model, Eckner (2009) also finds negative mean reverting rates of the default intensities. Besides, the calibrations of the Generalized-Poisson loss model performed by Brigo et al. (2007) also indicate upward sloping of expected default intensities. The explosive feature of default intensities suggests that investors take a more pessimistic view about the future default intensities. The volatility of the idiosyncratic intensity $\bar{\sigma}$ is about double that of the non-idiosyncratic intensities. Jump rates of the intensities ranges from two to ten per hundred years. Jump sizes of the intensities are moderate, ranging from 18 bps to 544 bps. These are significantly lower than the jump size found by Eckner (2009) which is around 3000 bps. The jump size in the correlated intensity model needs to be high in order to give enough default correlation among firms, while jumps of intensities in the stress event model have only a minor effect on the correlation among firms.

6. Conclusion

In this paper, we provide an efficient methodology to compute the loss distribution of a large portfolio in the stress event model. A new approach to independence conditions is proposed, which leads to significant simplifications in computing the loss distribution. We perform calibrations to market data and achieve good fits. In addition, the computational cost for additional common factors, unlike other bottom-up approaches, is mild.

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APPENDIX A

7. Basic Affine Jump Diffusions

A stochastic process λ_t on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_t, \mathrm{P})$ is called a basic Affine Jump Diffusion (AJD) if it satisfies the following SDE:

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dB_t + dJ_t, \tag{A1}$$

where B is a standard Brownian motion, and J is an independent compound Poisson process with jump intensity l and exponentially distributed jump sized with mean μ. Duffie et al. (2000) show that the moment generating function of a cumulative intensity $\Lambda(t) = \int_0^t \lambda_s ds$ admits a closed form solution

$$\mathrm{E}\left[e^{q\Lambda(t)}|\lambda_0\right] = e^{\alpha(t)+\beta(t)\lambda_0}, \tag{A2}$$

where

$$\alpha(t) = -\frac{2\kappa\theta}{\sigma^2}\log\left(\frac{c_1 + d_1 e^{-\gamma t}}{c_1 + d_1}\right) + \frac{\kappa\theta t}{c_1} \tag{A3}$$

$$+\, l\left(\frac{d_1/c_1 - d_2/c_2}{-\gamma d_2}\right)\log\left(\frac{c_2 + d_2 e^{-\gamma t}}{c_2 + d_2}\right) + \frac{l(1-c_2)t}{c_2} \tag{A4}$$

$$\beta(t) = \frac{1 - e^{-\gamma t}}{c_1 + d_1 e^{-\gamma t}}, \tag{A5}$$

and

$$\gamma = \sqrt{\kappa^2 - 2\sigma^2 q} \tag{A6}$$

$$c_1 = (\kappa + \gamma)/(2q) \tag{A7}$$

$$c_2 = 1 - \mu/c_1 \tag{A8}$$

$$d_1 = (-\kappa + \gamma)/(2q) \tag{A9}$$

$$d_2 = (d_1 + \mu)/c_1. \tag{A10}$$

With the help of the closed form expression of the moment generating function, we can compute the expectation

$$\mathrm{E}\left[e^{-\Lambda(t)}|\lambda_0\right], \tag{A11}$$

which is the probability that there is no stress event by time t, by plugging $q = -1$ in Eq.(A2)-Eq.(A10). Longstaff and Rajan (2008) derive a recursive system of ordinary differential equations to compute

$$\mathrm{E}\left[e^{-\Lambda(t)}(\Lambda(t))^k|\lambda_0\right]. \tag{A12}$$

Their approach is quite time consuming in solving the system of ODEs numerically. Besides, it is hard to control the error propagation in the recursive ODEs.

In fact, Eq.(A12) can be computed easily by differentiating Eq.(A2) k times with respect to q, then

$$\frac{d^k}{dq^k}\left(e^{\alpha(t)+\beta(t)\lambda_0}\right) = \mathrm{E}\left[e^{q\Lambda(t)}(\Lambda(t))^k|\lambda_0\right]. \tag{A13}$$

Plugging $q = -1$ and dividing by $k!$ yields the probability that there are k stress events in the sector, i.e.

$$\mathrm{P}(k \text{ stress events by time } t) = \frac{1}{k!}\frac{d^k}{dq^k}\left(e^{\alpha(t)+\beta(t)\lambda_0}\right)\Big|_{q=-1}. \tag{A14}$$

The validity of exchanging the order of differentiation and expectation in Eq.(A13) can be verified if $\Lambda(t) \geq 0$ for all t, which is true in our consideration here since $\Lambda(t)$, as a cumulative intensity, is always non-negative. As a result, in order to compute the scenario probability, $\mathrm{P}(k$ stress events by time $t)$, we just need to find the k-th derivative of the moment generating function Eq.(A2) at $q = -1$. Although Eq.(A14) admits a closed form expression, its complexity grows tremendously with k. For example, the closed form expression of Eq.(A14) for $k = 4$, obtained by the symbolic toolbox of MATLAB, needs 285 letter-size pages (with font size 12) to print the result. Therefore, evaluating Eq.(A14) can be quite time consuming even for moderate k and we need a more efficient way to calculate the derivatives. To this end, we adopt the exact numerical differentiation algorithm developed by Tsui (2010), which is very efficient in evaluating high order derivatives.

APPENDIX B

8. Determination of $\bar{\lambda}_i$ and p_i^l from CDS spreads

We will fix $\bar{\lambda}_i$ and p_i^l for each name of the portfolio by using the 5-year CDS spreads with the constraints

$$\bar{\lambda}_i \geq 0 \quad i = 1, ..., N, \tag{B1}$$

$$0 \leq p_i^l \leq, 1 \quad l = 1, 2, 3, \quad i = 1, ..., N. \tag{B2}$$

We start by defining a relative credit quality in terms of 5-year CDS spreads as follows:

$$c_i = \frac{S_i}{\frac{1}{N}\sum_{j=1}^{N} S_j}, \tag{B3}$$

Then, for $l = 1, 2$, define an auxiliary impact parameter

$$\tilde{p}_i^l = \min\{c_i p^l, 1\}, \quad i = 1, ..., N \tag{B4}$$

where p^l is a representative impact parameter of the l-sector which is to be calibrated to the tranche quotes. For $l = 3$, choose

$$0 \leq \tilde{p}_i^3 = p^3 \leq 1 \tag{B5}$$

for all i. For most of the situations, we can choose $p_i^l = \tilde{p}_i^l$. Recall that $\lambda_i = S_i/(1-R)$ and $\lambda^l \geq 0$ are parameters to be calibrated to the tranches, so the idiosyncratic default intensity is

$$\bar{\lambda}_i = \lambda_i - \tilde{p}_i^1\lambda^1 - \tilde{p}_i^2\lambda^2 - \tilde{p}_i^3\lambda^3. \tag{B6}$$

However, $\bar{\lambda}_i$ computed as above could be negative for some cases. For those cases, we lower the values of p_i^l proportionally, so

$$p_i^l = \begin{cases} \tilde{p}_i^l, & \text{if } \lambda_i - \tilde{p}_i^1\lambda^1 - \tilde{p}_i^2\lambda^2 - \tilde{p}_i^3\lambda^3 \geq 0; \\ \dfrac{\lambda_i \tilde{p}_i^l}{\tilde{p}_i^1\lambda^1 + \tilde{p}_i^2\lambda^2 + \tilde{p}_i^3\lambda^3}, & \text{otherwise,} \end{cases} \tag{B7}$$

for all l and i, and

$$\bar{\lambda}_i = \lambda_i - p_i^1\lambda^1 - p_i^2\lambda^2 - p_i^3\lambda^3. \tag{B8}$$

With a fixed set of parameters

$$\Theta = \{\lambda^1, p^1, \lambda^2, p^2, \lambda^3, p^3\}, \tag{B9}$$

the CDS spreads S_i can be matched exactly by choosing p_i^l and $\bar{\lambda}_i$ by Eq.(B7) and Eq.(B8) respectively. For $l = 1, 2$, the specification of p_i^l basically follows the idea of Eckner (2009) where the dependence on a factor is proportional to the relative credit quality c_i. For $l = 3$, we choose p_i^3 to be the same if possible to include the possibility of some catastrophic events that have a high probability to kill many firms.

REFERENCES

Andersen, L., J. Sidenius, and S. Basu, 2003: All your hedges in one basket. *Risk*, **November**.

Arnsdorf, M. and I. Halperin, 2008: Bslp: Markovian bivariate spread-loss model for portfolio credit derivatives. *Journal of Computational Finance*, **12**, 77–100.

Bayraktar, E. and B. Yang, 2009: Multi-scale time-changed birth processes for pricing multi-name credit derivatives. *Applied Mathematical Finance*, **16(5)**, 429–449.

Brigo, D., A. Pallavicini, and R. Torresetti, 2007: CDO calibration with the dynamical generalized Poisson loss model. *Risk Magazine*, **June**.

Cont, R. and A. Minca, 2008: Extracting portfolio default rates from cdo spreads. Working paper, Columbia University.

Duffie, D. and N. Gârleanu, 2001: Risk and valuation of collateralized debt obligations. *Financial Analysts Journal*, **57**, 41–59.

Duffie, D., J. Pan, and J. Singleton, 2000: Transform analysis and asset pricing for affine jump diffusions. *Econometrica*, **68, No. 6 (November)**.

Duffie, D. and K. Singleton, 1999: Simulating correlated defaults, graduate School of Business, Stanford Univeristy.

Eckner, A., 2009: Computational techniques for basic affine models of portfolio credit risk. *Journal of Computational Finance*, **13**.

Errais, E., K. Giesecke, and L. Goldberg, 2006: Pricing credit from top down with affine point processes. Working paper, Stanford University.

Giesecke, K., L. Goldberg, and X. Ding, 2010: A top-down approach to multi-name credit. *Operations Research*, **forthcoming**.

Hamilton, D., P. Varma, S. Ou, and R. Cantor, 2004: Default and recovery rates of corporate bond issuers. *Moody's Investors Service, New York*, **January**.

Joshi, M. and A. Stacey, 2006: Intensity gamma: A new approach to pricing portfolio credit derivatives. *RISK*, **19**, July 78–83.

Longstaff, F. and A. Rajan, 2008: An empirical analysis of the pricing of collateralized debt obligations. *Journal of Finance*, **63**, 529–563.

Mortensen, A., 2006: Semi-analytical valuation of basket credit derivatives in intensity-based models. *Journal of Derivatives*, **13, No. 4**, 8–26.

O'Kane, D., 2008: *Modelling Single-name and Multi-name Credit Derivatives*. The Wiley Finance Series.

Papageorgiou, E. and R. Sircar, 2007: Multiscale intensity models and name grouping for valuation of multi-name credit derivatives. *Applied Mathematical Finance*, **15(1)**, 73–105.

Peng, X. and S. S. G. Kou, 2008: Default clustering and valuation of collateralized debt obligations. Working paper, Columbia University.

Schönbucher, P., 2003: *Credit Derivatives Pricing Models*. John Wiley & Son Ltd.

Shelton, D., 2004: Back to normal, proxy integration: A fast accurate method for cdo and cdo-squared pricing. *Citigroup Structured Credit Research*, **August**.

Tsui, L. K., 2010: Exact numerical algorithm for n-th order derivative of a single variable function. Working paper, University of Pittsburgh.

List of Tables

TABLE 1. Number of conditional loss distributions required for K-th order approximation with L sectors.

$K \setminus L$	1	2	3	4	5
1	2	3	4	5	6
2	3	6	10	15	21
3	4	10	20	35	56
4	5	15	35	70	126
5	6	21	56	126	252
6	7	28	84	210	462
7	8	36	120	330	792
8	9	45	165	495	1287

Table 2. Tranche quotes of CDX.NA.IG series 13 on April 15 2010. All quotes are upfronts in percentage with fixed 100bps running spread.

CDX	0-3%	$3-7\%$	$7-10\%$	$10-15\%$	$15-30\%$
Bid	51.530	16.000	4.888	-1.210	-3.100
Mid	52.185	16.605	5.345	-0.855	-2.880
Ask	52.840	17.210	5.810	-0.500	-2.660

TABLE 3. Summary of the closing data of the mid 5-year CDS spreads of the 125 names in CDX.NA.IG series 13 on April 15 2010.

Statistics	bps
Min	25.39
Max	349.62
Median	74.36
Mean	87.76
Standard Deviation	47.04

Table 4. Model parameters calibrated to tranche spreads of CDX.NA.IG series 13 and the underlying CDS spreads.

	λ^1	p^1	λ^2	p^2	λ^3	p^3
CDX	0.0429520	0.0875257	0.0130048	0.1226815	0.0037735	1.000000

TABLE 5. Model implied tranche spreads of CDX.NA.IG series 13 on April 15 2010 using fifth order calculation. Bid-ask spreads are included for comparison.

CDX	0-3%	$3-7\%$	$7-10\%$	$10-15\%$	$15-30\%$
Bid	51.530	16.000	4.888	-1.210	-3.100
Model	52.265	16.695	5.383	-0.838	-2.686
Ask	52.840	17.210	5.810	-0.500	-2.660

TABLE 6. Market mid prices of Markit iTraxx Europe series 7 version 1. 0-3% tranche is quoted in percentage as an upfront with a fixed 500bps running spread and all the other tranches are spreads in bps without upfront.

Maturity	Tranche	Mar 30, 07	Apr 30, 07	May 31, 07	Jun 29, 07
5-year	0-3%	11.23%	9.94%	6.33%	11.75%
	3-6%	57.75	49.82	39.90	62.05
	6-9%	14.28	12.45	10.33	16.29
	9-12%	6.24	5.53	4.39	7.48
	12-22%	2.58	2.54	1.93	3.10
7-year	0-3%	25.77%	24.84%	20.61%	26.38%
	3-6%	133.79	121.2	105.08	137.13
	6-9%	37.25	31.99	27.04	37.39
	9-12%	17.33	15.75	13.05	17.00
	12-22%	5.85	5.67	5.20	7.50
10-year	0-3%	40.51%	38.95%	35.00%	40.53%
	3-6%	338.96	322.20	294.21	368.60
	6-9%	98.59	93.48	85.17	108.55
	9-12%	46.91	43.59	38.98	50.33
	12-22%	14.38	14.50	12.20	15.95

Table 7. Model parameters with $K = 8$ calibrated to tranche spreads of Markit iTraxx Europe series 7 version 1.

$\overline{\kappa}$	$\overline{\sigma}$	$\bar{l}$	$\overline{\mu}$
-0.19135	0.17548	0.09215	0.00761

κ^1	σ^1	l^1	μ^1	p^1
-0.10154	0.20062	0.13031	0.05438	0.03269

κ^2	σ^2	l^2	μ^2	p^2
-0.68412	0.17139	0.02529	0.001758	0.2256

	Mar 30, 07	Apr 30, 07	May 31, 07	Jun 29, 07
$\overline{\lambda}_0$	0.00044170	0.00044276	0.00002512	0.00055822
λ_0^1	0.00366764	0.00143970	0.00000000	0.00631753
λ_0^2	0.00002524	0.00002555	0.00001003	0.00006193

Table 8. Model implied tranche prices of Markit iTraxx Europe series 7 version 1.

Maturity	Tranche	Mar 30, 07	Apr 30, 07	May 31, 07	Jun 29, 07
5-year	0-3%	11.00%	9.79%	6.26%	11.53%
	3-6%	58.85	50.82	42.16	62.76
	6-9%	13.84	12.30	10.16	14.84
	9-12%	6.18	5.79	4.72	7.26
	12-22%	2.57	2.45	1.90	3.24
7-year	0-3%	26.11%	24.89%	20.66%	26.85%
	3-6%	132.35	118.96	99.969	139.45
	6-9%	37.37	33.41	28.58	39.57
	9-12%	15.85	14.56	12.47	17.24
	12-22%	6.11	5.86	4.91	7.31
10-year	0-3%	42.46%	41.77%	38.64%	43.63%
	3-6%	334.12	316.78	277.15	345.16
	6-9%	101.01	92.66	81.75	106.38
	9-12%	47.17	43.16	38.32	49.81
	12-22%	14.92	14.00	12.51	16.06